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THE GENERALIZED-PLANE PROBLEM OF THE THEORY OF ELASTICITY ON THE ROTATION OF A RIGHT PRISM WITH A SQUARE CROSS-SECTION†

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The problem of the generalized-plane deformation of a rotating long square elastic prism is solved. The ends of the prism are load free. A modification of Mathieu's method for constructing the solution of this problem with mass forces is proposed. The proposed modification involves the use of the polynomial solutions of a biharmonic equation in addition to ordinary Fourier series. The existence of these polynomial solutions enables one to increase the convergence of the Fourier series substantially. The stressed state of the prism and the distortion of its faces are investigated.

The method of double trigonometric series has been proposed [1] for solving the plane problem in the theory of elasticity for a rectangle with arbitrary loading of the edges of rectangles and when there are mass forces. The weak convergence of the series is a drawback of this method. Another approach to investigating the plane problem in the theory of elasticity uses a superpositioning of ordinary Fourier series along one or other of the coordinates (Mathieu's method) where each term of these series satisfies a biharmonic equation. A review of the papers associated with the development of this method for solving plane problems is given in [2].

1. FORMULATION OF THE PROBLEM AND THE METHOD OF SOLUTION

A long isotropic elastic prism with a square cross-section is rotated at a constant angular velocity ω around the z axis which passes through the centre of its cross-sections. The ends of the prism are load free. We shall solve the problem of the generalized-plane deformation of this prism in a rectangular system of dimensionless coordinates x, y referred to the half-length of the side of the square with its origin at the centre of the square cross-section. The x and y axes are directed parallel to the sides of the square and the lateral faces of the prism are the surfaces $x = \pm 1$ and $y = \pm 1$, respectively. The equilibrium equation and the equation for the compatibility of the strains which describe the generalized-plane deformation of the prism under investigation have the form

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+xP=0,\quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+yP=0$$
$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2}=\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y},\quad E\varepsilon_{zz}=C_z,\quad P=\rho\omega^2l^2 \tag{1.1}$$

We will write the relation between the stresses and the strains in the form

$$E(1-\nu^2)^{-1}\varepsilon_{xx}=\sigma_{xx}-\nu(1-\nu)^{-1}\sigma_{yy}-\nu(1-\nu^2)^{-1}C_z$$
$$E(1-\nu^2)^{-1}\varepsilon_{yy}=\sigma_{yy}-\nu(1-\nu)^{-1}\sigma_{xx}-\nu(1-\nu^2)^{-1}C_z \tag{1.2}$$
$$E(1-\nu^2)^{-1}\varepsilon_{xy}=2(1-\nu)^{-1}\sigma_{xy},\quad E\varepsilon_{zz}=\sigma_{zz}-\nu(\sigma_{xx}+\sigma_{yy})$$

Here E is Young's modulus, ν is Poisson's ratio, ρ is the density of the material, ω is the angular velocity of rotation of the prism and l is half the length of the sides of the square cross-section of the prism.

The constant C_z is determined from the condition that the resultant force on the ends of the prism is equal to zero

$$\int_{-1}^{1}\int_{-1}^{1}\sigma_{zz}dxdy=0 \tag{1.3}$$

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We will write the boundary conditions on the lateral surfaces of the prism as

$$\sigma_{xx} = \sigma_{xy} = 0, \quad x = \pm 1; \quad \sigma_{yy} = \sigma_{xy} = 0, \quad y = \pm 1 \tag{1.4}$$

Taking account of the symmetry of the problem, the exact solution of Eqs (1.1) and (1.2) can be written in the form

$$\begin{aligned}
&\sigma_{xx} = (x,y) = -\frac{P(x^2-1)}{2(1-\nu)} + D_1(x^2-1-y^2) + \\
&+D_2(6x^2y^2 - 3y^4 + x^4 - 1) + M + \sum_{n=1}^{\infty} [f_n(x,y) - \varphi_n(x,y)] \\
&f_n(x,y) = \cos(\alpha_n x)\{C_{1,n}\operatorname{ch}(\alpha_n y) + C_{2,n}[2\alpha_n^{-1}\operatorname{ch}(\alpha_n y) + y\operatorname{sh}(\alpha_n y)]\}, \\
&\alpha_n = \pi n, \quad n \geqslant 1 \\
&\varphi_n(x,y) = \cos(\alpha_n y)\{C_{1,n}\operatorname{ch}(\alpha_n x) + C_{2,n} x \operatorname{sh}(\alpha_n x)\} \\
&\sigma_{xy}(x,y) = \frac{\nu x y P}{1-\nu} - 2xyD_1 - D_2 4(xy^3 + x^3 y) + \sum_{n=1}^{\infty} [\psi_n(x,y) + \psi_n(y,x)] \\
&\psi_n(x,y) = \sin(\alpha_n x)\{C_{1,n}\operatorname{sh}(\alpha_n y) + C_{2,n}[\alpha_n^{-1}\operatorname{sh}(\alpha_n y) + y\operatorname{ch}(\alpha_n y)]\}
\end{aligned} \tag{1.5}$$

Here D_1, D_2, $C_{1,n}$, $C_{2,n}$ are constants which are to be determined during the solution. The expression for $\sigma_{yy}(x, y)$ is obtained from the formula for calculating $\sigma_{yy}(x, y)$ by the cyclic substitution $(x, y) \to (y, x)$.

In formulae (1.5), the first term (which is proportional to P) defines a particular solution of Eqs (1.1) and (1.2) with mass forces. The other terms satisfy the homogeneous system of equations (1.1) and (1.2) which, as is well known, reduces to a biharmonic equation.

We put

$$D_1 + 4D_2 = \frac{\nu P}{2(1-\nu)} \tag{1.6}$$

In this case, the expression σ_{xy} contains the polynomial $-4D_2(x^3y + xy^3 - 2xy)$ in addition to Fourier series.

By virtue of the symmetry of relations (1.5), it is sufficient to satisfy the boundary conditions on the lateral surface $x = 1$. The remaining conditions (1.4) and, also, the condition that the vector of the moment acting in cross-sections of the prism is equal to zero are automatically satisfied.

We now expand the hyperbolic functions in Fourier series in the following manner

$$\begin{aligned}
&\operatorname{ch}(\alpha_n y) = \frac{1}{2} y^2 \alpha_n \operatorname{sh}\alpha_n + \sum_{m=0}^{\infty} b_{n,m}^1 \cos(\alpha_m y) \\
&y\operatorname{sh}(\alpha_n y) = \frac{1}{2} y^2 (\alpha_n \operatorname{ch}\alpha_n + \operatorname{sh}\alpha_n) + \sum_{m=0}^{\infty} b_{n,m}^2 \cos(\alpha_m y)
\end{aligned} \tag{1.7}$$

The order of the asymptotic decay of the coefficients $b^1_{n,m}$, $b^2_{n,m}$ with respect to m is equal to m^{-4}.

When the formulae

$$y - y^3 = \sum_{m=1}^{\infty} s_m \sin(\alpha_m y), \quad \alpha_m = \pi m, \quad s_m = -12(-1)^m (\pi m)^{-3} \tag{1.8}$$

$$2y^2 - y^4 = \frac{7}{15} + 48 \sum_{m=1}^{\infty} (-1)^m (\pi m)^{-4} \cos(\alpha_m y)$$

are taken into account, we conclude that the equation $\sigma_{xy}(1, y) = 0$ reduces to the system

$$4D_2 s_m + C_{1,m}\operatorname{sh}\alpha_m + C_{2,m}(\alpha_m^{-1}\operatorname{sh}\alpha_m + \operatorname{ch}\alpha_m) = 0, \quad m = 1,2,\ldots \tag{1.9}$$

In the equation $\sigma_{xx}(1, y) = 0$, we expand the hyperbolic functions in Fourier series and, also, the term $3D_2(2y^2 - y^4)$ in accordance with formulae (1.7) and (1.8).

As a result, we obtain the system of algebraic equations

$$\begin{aligned}
&D_2 144(-1)^m \alpha_m^{-4} + \sum_{n=1}^{\infty} (-1)^n \{C_{1,n} b_{n,m}^1 + C_{2,n}(2\alpha_n^{-1} b_{n,m}^1 + b_{n,m}^2)\} - \\
&-C_{1,m}\operatorname{ch}\alpha_m - C_{2,m}\operatorname{sh}\alpha_m = 0, \quad m = 1,2,\ldots
\end{aligned} \tag{1.10}$$

$$D_1 = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\{C_{1,n}\alpha_n \operatorname{sh}\alpha_n + C_{2,n}(3\operatorname{sh}\alpha_n + \alpha_n \operatorname{ch}\alpha_n)\}$$
$$-M = \frac{7}{5}D_2 + \sum_{n=1}^{\infty}(-1)^n\{C_{1,n}b^1_{n,0} + C_{2,n}(2\alpha_n^{-1}b^1_{n,0} + b^2_{n,0})\} \tag{1.11}$$

Equation (1.6) closes the system of infinite algebraic equations for determining the constants $C_{1,m}$, $C_{2,m}$, D_1, D_2 and M. An expansion of the functional equation $\sigma_{xx}(1, y) = 0$ in the basis functions y^2, 1, $\cos(\pi m y)$ was used in deriving the system of equations (1.10) and (1.11).

Note that the convergence of the ordinary Fourier series (1.5) is faster, the greater the order of the asymptotic decay of the free terms (which are proportional to D_2 with respect to m and the coefficients $b^1{}_{n,m}$, $b^2{}_{n,m}$ of the infinite system of equations (1.10) and, also, the free terms ($4D_2s_m$) in system (1.9). In the given case (when there are two polynomials with indefinite constants D_1 and D_2 in the solution (1.5)), the order of decay of the free terms and of the coefficients $b^1{}_{n,m}$, $b^2{}_{n,m}$ in system (1.1) is equal to m^{-4}. In system (1.9), the order of the decay of the free terms is equal to m^{-3}. Calculations show that the solution (1.5), (1.6), (1.9)–(1.11) which has been constructed possesses a very high convergence.

The constant C_2 is determined from condition (1.3).

The dimensionless displacements (divided by l) are calculated using the formulae

$$u_x^*(x,y) = \int_0^x \varepsilon_{xx}(\eta, y)d\eta, \quad u_y^*(x,y) = \int_0^y \varepsilon_{yy}(x,\xi)d\xi$$

2. RESULTS OF CALCULATIONS

It was noted in the preceding section that, when $\nu = 0$, the solution of the problem has the form

$$\sigma_{xx} = -\frac{1}{2}P(x^2 - 1), \quad \sigma_{yy} = -\frac{1}{2}P(y^2 - 1), \quad \sigma_{xy} = \sigma_{zz} = 0$$

In this case, the faces of the prism $x = \pm 1, y = \pm 1$ remain plane when the prism is deformed.

It is obvious that the shear stresses in the prism under investigation depend very much on Poisson's ratio ν.

Below, we present the values of the dimensionless stresses (divided by P) at certain characteristic points of the cross-section of the prism when $\nu = 0.5$ and $L = 50$ (the number of the term at which the sums of the Fourier series are formed)

$(x; y)$	(0; 0)	(0; 1)	(0.4; 0.4)	(0.6; 0.6)	(1; 0.9)	(1; 1)
$\sigma^*_{xx} \times 10^4$	6410	893	4909	3312	0	1
$\sigma^*_{yy} \times 10^4$	6410	0	4909	3312	817	1
$\sigma^*_{xy} \times 10^5$	0	0	6466	9707	1	0
$\sigma^*_{zz} \times 10^4$	3076	−2887	1575	−21	−2925	−3332

Note that, when $L \geqslant 15$, the results are practically identical.

We now present the results of a calculation of the stresses in the case of a square prism (l = 10 mm) made of beryllium [3] ($\nu = 0.02$, $E = 3 \times 10^{11}$ N/m^2, $\rho = 1.8$ g/cm^2) which rotates at an angular velocity $\omega = 10^4\pi$ s^{-1}

$(x; y)$	(0; 0)	(0; 1)	(0.6; 0.6)	(1; 1)
$\sigma^*_{xx} \times 10^4$	5029	4916	3202	0
$\sigma^*_{yy} \times 10^4$	5029	0	3202	0
$\sigma^*_{xy} \times 10^4$	0	0	19.81	0
$\sigma^*_{zz} \times 10^4$	67.82	−35.01	−5.24	−133.3

as well as the dimensional values of the displacements $u_x(1, y)$ and $u_y(1, y)$ of the face $x = 1$

y	0	0.25	0.5	0.75	1
u_x, μm	1.96	1.96	1.97	1.98	1.98
u_y, μm	0	0.73	1.36	1.81	1.98

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